

Small holding circles

Augustin FRUCHARD

*Laboratoire MIA, EA3993, Université de Haute Alsace,
4, rue des Frères Lumière, F-68093 Mulhouse cedex, France*

E-mail: Augustin.Fruchard@uha.fr

Tel. (+33) 389 33 66 37

Fax (+33) 389 33 66 53

Abstract: A circle C holds a convex body K if C does not meet the interior of K and if there does not exist any euclidean displacement which moves C as far as desired from K , avoiding the interior of K . The purpose of this note is to explore how small can be a holding circle. In particular it is shown that the diameter of such a holding circle can be less than the width w of the body but is always greater than $2w/3$.

Keywords: holding circle, immobilization, width, diameter, convex body.

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1 Introduction

The question of holding a convex body has been often considered in the literature. In [8], Coxeter asked about the minimal total length of edges of a cage holding the unit ball. Besicovitch [4] and Valette [24] investigated this question. Analogous problems of caging are still widely studied, see e.g. [20, 21, 23]. Concerning circumscribing polyhedra, Besicovitch and Eggleton show in [2] that the polyhedron with minimal total length of edges enclosing the unit ball is a cube. In [3], Besicovitch determines the minimal length of a net holding the unit ball.

F. Caragiù asked whether convex bodies exist that can be held by a very simple instrument such as a circle. T. Zamfirescu gives the answer in [25]: not only such convex bodies exist, but they form a huge majority. More precisely, they form a subset with dense interior among all convex bodies, with respect to the Hausdorff-Pompeiu distance. Whether this subset is open or not is unclear up to my knowledge.

In the whole article, K denotes a convex body of \mathbb{R}^3 which admits holding circles; to shorten we say a *holdable convex body*. Let w denote the width of K (i.e. the minimal distance between two parallel planes enclosing the body), D the minimal diameter of a circumscribing cylinder and d the minimal diameter of a holding circle. We will also consider the supremum, denoted δ , of diameters of holding circles (in the article, the word “diameter” refers sometimes to a segment, sometimes to its length).

Several quantities may measure how large or small are holding circles: e.g. the ratios $\frac{d}{D}$, $\frac{d}{w}$, $\frac{\delta}{D}$ and $\frac{\delta}{w}$. Of course we have $\frac{d}{D} \leq \frac{d}{w} \leq \frac{\delta}{w}$ and $\frac{d}{D} \leq \frac{\delta}{D} \leq \frac{\delta}{w}$. In Section 3.7, we provide an example, due to T. Zamfirescu, where all these ratios are as large as desired. Section 3.8

contains examples with ratios $\frac{d}{D}$ and $\frac{\delta}{D}$ as small as desired. Concerning the ratio $\frac{d}{w}$, our main result is the following.

Theorem 1 . (a) *For any holdable convex body K , we have $\frac{d}{w} > \frac{2}{3}$.*

(b) *The constant $\frac{2}{3}$ is sharp: there are convex bodies with $\frac{d}{w}$ as close to $\frac{2}{3}$ as desired.*

The surprising Item (b) gives a negative answer to a question of Joël Rouyer, who wondered if d would always be greater than or equal to w . See Figure 3 for a family of bodies satisfying (b). The same result holds for the ratio $\frac{\delta}{w}$, see Section 3.6. In Section 3 we also discuss the higher dimensional case (Section 3.1), the case of few-vertex polyhedra (3.2 and 3.3), the cases of “asymptotic” equality (3.5) and related topics in the literature (3.9).

2 Proof of Theorem 1

We first have to deal briefly with an aspect of planar convexity. Given a non-horizontal strip in \mathbb{R}^2

$$S = \{(x, y) \in \mathbb{R}^2 ; ay + b_1 \leq x \leq ay + b_2\}$$

the *horizontal width* of S is $w_h(S) = b_2 - b_1$. Given a convex compact subset B of \mathbb{R}^2 , the *horizontal width* of B , $w_h(B)$, is the infimum of $w_h(S)$ over all strips S containing B , see Figure 1. Of course, we have $w_h(B) \geq w_2(B)$, the usual planar width of B .

Lemma 2 . *Let A, B be convex compact subsets of \mathbb{R}^2 , A in the closed upper half-plane and B in the closed lower half-plane, such that $A \cup B$ is also convex. Then we have*

$$w_h(A \cap B) = \min(w_h(A), w_h(B)) \text{ and } w_h(A \cup B) = \max(w_h(A), w_h(B)).$$

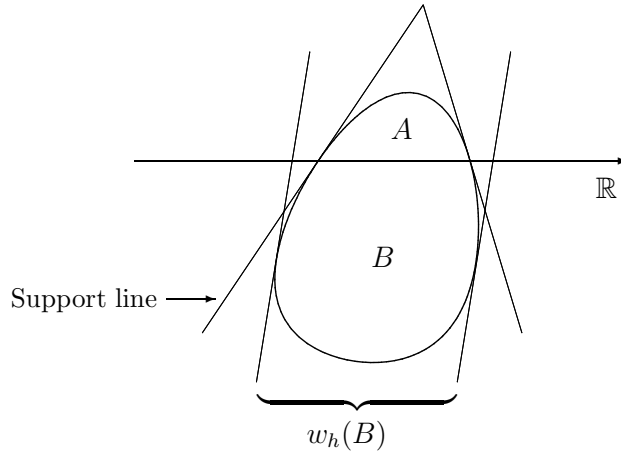


Figure 1: The subsets A, B , two support lines of $A \cup B$ intersecting $A \cap B$ and a strip of minimal horizontal width containing $A \cup B$.

Proof of Lemma 2. Observe that $A \cap B$ is a segment of $\mathbb{R} \times \{0\}$. We obviously have $w_h(A \cap B) \leq \min(w_h(A), w_h(B))$ and $w_h(A \cup B) \geq \max(w_h(A), w_h(B))$. In order to prove that these are equalities, consider two support lines of $A \cup B$ at each end of $A \cap B$, see Figure 1. If these lines can be chosen parallel, then we have $w_h(A \cap B) = w_h(A \cup B)$. Otherwise they cross, say above, and then A is included in a triangle of horizontal width $w_h(A \cap B)$. This proves the first equality. For the second one, consider a strip of minimal horizontal length containing B . The boundary of this strip contains (at least) two points of the boundary of B at the same altitude

on each side, and which are not in $A \cap B$ (remind that we are in the case where there are no parallel support lines of $A \cup B$ at each end of $A \cap B$). Therefore such a strip must contain A , yielding $w_h(B) = w_h(A \cup B)$. \square

We now fix a holdable convex body K and a holding circle C of minimal diameter d in a horizontal position.

Let H denote the plane containing C . This plane cuts K in two convex bodies: K^+ above and K^- below. Let K_0 denote their intersection: $K_0 = K^+ \cap K^- = K \cap H$. We call a *horizontal slice* of K^+ the intersection of K^+ with a horizontal plane.

For any $\theta \in [0, \pi[$, let V_θ denote the vertical plane making an angle θ with $0x$. Let A_θ , resp. B_θ denote the orthogonal projection of K^+ , resp. K^- in V_θ . Notice that the orthogonal projection of the holding circle, which is a segment of length d , must contain the segment $A_\theta \cap B_\theta$.

Definition 3 . With the above notation, we call K an *iceberg* if there exists a holding circle and an orientation of its axis such that $w_h(A_\theta) < w_h(B_\theta)$ for all $\theta \in [0, \pi[$.

At a first glance it seems that icebergs cannot exist: if A_θ is narrower than B_θ for any direction θ , then it seems that the circle could be released through A . Nevertheless, as shows Figure 3, icebergs do exist. In Section 3.3, we prove that tetrahedra and five-vertex polyhedra cannot be icebergs.

Notice that the width w of K satisfies $w \leq w_2(A_\theta \cup B_\theta) \leq w_h(A_\theta \cup B_\theta)$ for all $\theta \in [0, \pi[$. It follows from Lemma 2 that, if K is an iceberg, then $w \leq w_h(B_\theta)$ for all $\theta \in [0, \pi[$. Notice also that, if $w_h(B_\theta) < w_h(A_\theta)$ for all $\theta \in [0, \pi[$, then K is an iceberg: it suffices to change the orientation of C .

Proof of Theorem 1 .

(a) Two cases occur. Firstly, if K is not an iceberg, this means that for some values of θ we have $w_h(A_\theta) \leq w_h(B_\theta)$ and for some other ones we have $w_h(A_\theta) \geq w_h(B_\theta)$. By continuity, there is a value $\theta_0 \in [0, \pi[$ for which $w_h(A_{\theta_0}) = w_h(B_{\theta_0})$. Then we obtain from Lemma 2

$$w \leq w_h(A_{\theta_0} \cup B_{\theta_0}) = w_h(A_{\theta_0} \cap B_{\theta_0}) \leq d. \quad (1)$$

By the way, this shows that, if $d < w$ then K is necessarily an iceberg.

Secondly, if K is an iceberg with $\frac{d}{w} < 1$ (otherwise there is nothing to prove), then we have $d < w \leq w_h(B_\theta)$ for all θ . Since C holds K , there is a horizontal slice K_h of K^+ whose circumscribing circle C_h has a diameter d_h larger than d , otherwise C could be released by a translation along the (continuous) curve of circumscribing centers of horizontal slices. Let Δ denote the straight line joining the centers of C and C_h . Let Π denote the — a priori non orthogonal — projection of direction Δ into H (we recall that H is the horizontal plane containing C). Let φ denote the homothety of center the center of C and of ratio $\frac{d}{d_h}$; in this manner, we have $C = \varphi(\Pi(C_h))$. Given $a \in C_h$, let P_a denote the plane tangent to C_h at a and parallel to Δ and set $P'_a = \varphi(P_a)$; hence P'_a is a plane parallel to Δ and tangent to C at $\varphi(\Pi(a))$, see Figure 2.

For each $a \in C_h \cap K_h$, consider the cone of vertex a and generatrix C . This cone contains K^- because two points of K^+ and K^- are joined by a segment which crosses the disk of boundary C . It follows that the closed half-space, denoted by E_a , containing C and delimited by P'_a contains $K^- \setminus K_0$ in its interior. This means that $K^- \setminus K_0$ is in the interior of the intersection, denoted by I , of the half-spaces E_a for all $a \in C_h \cap K_h$. Since $d < w_h(B_\theta)$ for all θ , the width $w_h(B_\theta)$ is not attained close to the plane H in the following sense: there exists $\varepsilon = \varepsilon(\theta) > 0$

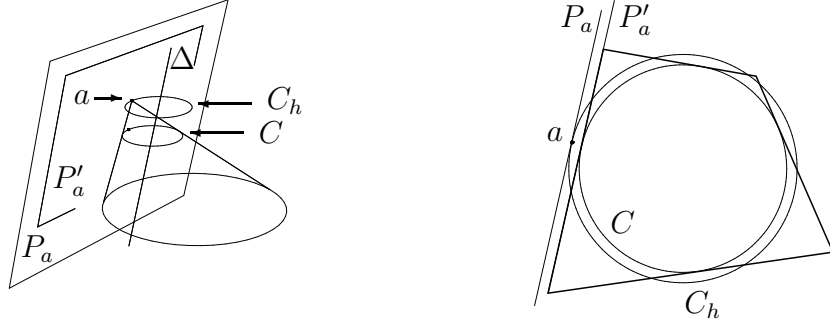


Figure 2: On the *left*, the two circles and the straight line Δ joining their centers, the cone with vertex $a \in C_h \cap K_h$ and the two planes P_a and P'_a . On the *right*, the images of P_a , P'_a , C and C_h by Π ; in **bold**, the boundary of $I \cap H$.

such that $w_h(B_\theta) = w_h(B_\theta \cap \{z \leq -\varepsilon\})$ (in fact, by compactness a single ε is available for all θ , but this is not necessary). Since $K^- \cap \{z \leq -\varepsilon\}$ is a compact subset of the interior of I , we obtain $w_h(B_\theta) < w_h(I_\theta)$ for all $\theta \in [0, \pi[$, where I_θ denotes the orthogonal projection of I into V_θ . The functions $\theta \mapsto w_h(B_\theta)$ and $\theta \mapsto w_h(I_\theta)$ are continuous on the compact set $[0, \pi]$, hence reach their infimum. Since Π maps I into $I \cap H$, we have $\min_{\theta \in [0, \pi[} w_h(I_\theta) = w_2(I \cap H)$, the planar width of $I \cap H$.

Because the circumscribing circle of $C_h \cap K_h$ is C_h itself, the circumscribing circle of the points $\varphi(\Pi(a))$, $a \in C_h \cap K_h$ is C itself. Therefore C is the greatest circle inscribed in $I \cap H$, hence satisfies $d \geq \frac{2}{3}w_2(I \cap H)$, as is well-known since Blaschke [5]. To sum up, we have

$$w \leq \min_{\theta \in [0, \pi[} w_h(A_\theta \cup B_\theta) = \min_{\theta \in [0, \pi[} w_h(B_\theta) < \min_{\theta \in [0, \pi[} w_h(I_\theta) = w_2(I \cap H) \leq \frac{3d}{2}. \quad (2)$$

(b) With the identification $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ and the notation $j = \exp(\frac{2\pi i}{3})$, choose $a > 1$ (close to 1) and $h > 0$ (large), and consider K the octahedron with vertices $(a, 0)$, $(ja, 0)$, $(j^2a, 0)$, $(-2, -h)$, $(-2j, -h)$, $(-2j^2, -h)$. It has a horizontal holding circle C of diameter $d = 2a \cos \varphi$, where $\varphi = \arctan(\frac{a-1}{\sqrt{3}})$, and of center $(0, -\frac{ha}{2\sqrt{3}} \sin(2\varphi))$. If a tends to 1 then d tends to 2 and if h tends to infinity, then w tends to 3. Hence the ratio $\frac{d}{w}$ is as close to $\frac{2}{3}$ as desired. Notice that the orthogonal projection in the horizontal plane shows K as a hexagon close to an equilateral triangle and C as its largest inscribed circle, see Figure 3. Observe also that three of the lateral faces are almost vertical.

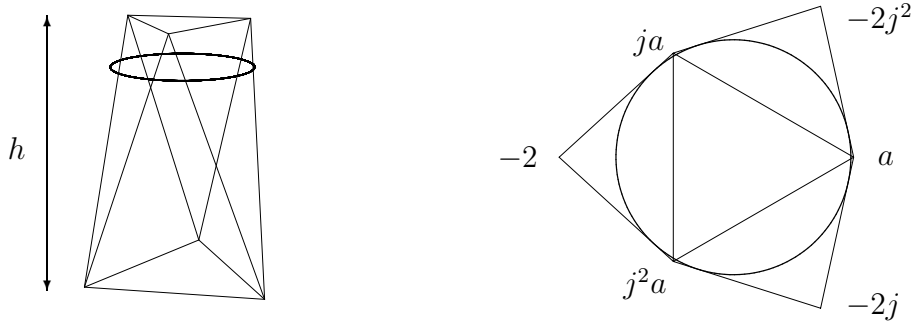


Figure 3: an octahedral iceberg and its smallest *holding circle*; here $a = 1.38$ and $h = -5$.

3 Remarks and examples

1. As suggested by the referee, Theorem 1 and its proof can be generalized in arbitrary dimension $n \geq 3$: if a convex body $K \subset \mathbb{R}^n$ of width w admits a holding sphere of dimension $n - 2$ and diameter d , then necessarily $\frac{d}{w} > C(n)$ where $C(n) = \frac{1}{\sqrt{n-1}}$ if n is even and $C(n) = \frac{\sqrt{n+1}}{n}$ if n is odd. The proof is the same, replacing the word “circle” by “ $(n - 2)$ -sphere” and “plane” by “hyperplane”, and using the Steinhagen inequality [22]: the width $w_{n-1}(K)$ and the inradius $r_{n-1}(K)$ of a convex body $K \subset \mathbb{R}^{n-1}$ satisfy $\frac{r_{n-1}(K)}{w_{n-1}(K)} \geq \frac{C(n)}{2}$, see e.g. [11] pp. 112–114 for a short proof.

To see that the constant $C(n)$ is sharp, we consider \mathbb{R}^n euclidean with coordinates x_1, \dots, x_n and split it in $\mathbb{R}^{n-1} \times \mathbb{R}$. In \mathbb{R}^{n-1} , consider the regular simplex, denoted S_a , centered at the origin and with a vertex at $(a, 0, \dots, 0)$, where $a > 1$ is close to 1. Its side-length is $a\sqrt{\frac{2n}{n-1}}$. Put this simplex in the hyperplane $x_n = 0$; this is denoted $S_a \times \{0\}$. With the same notation, consider the simplex S_{1-n} and put it in the hyperplane $x_n = -h$ with $h > 0$ large. Then the convex hull of $(S_a \times \{0\}) \cup (S_{1-n} \times \{-h\})$ has a width close to $\frac{2}{C(n)}$ and a holding $(n - 2)$ -sphere of diameter $2a \left(1 + \frac{(a-1)^2}{n^2-2n}\right)^{-1/2}$, hence close to 2.

2. There is a short proof that $w \leq d$ for tetrahedra: if K is a polyhedron with a holding circle C of minimal diameter d , then C contains at least two points of K on two non-intersecting edges, hence the diameter of C is at least the distance between these edges, i.e. the distance between the two parallel planes that contain each of these edges. In the case of a tetrahedron, all the vertices, hence the whole tetrahedron, are in the closed spatial strip between these planes.

3. The example in Figure 2 has the minimal number of vertices required for an iceberg, because neither tetrahedra nor five-vertex polyhedra can be icebergs. Actually, assume by contradiction that a five-vertex polyhedron is an iceberg (the proof is similar for a tetrahedron). With the notation above Definition 3, among K^+ and K^- , one contains at most two vertices of K , say K^+ . Let the five vertices be labelled $a, b \in K^+$ and $c, e, f \in K^-$ and let θ_1 be such that the vertical plane V_{θ_1} contains the direction of the edge ab . Then in this direction we have $w_h(B_{\theta_1}) \leq w_h(A_{\theta_1})$, because otherwise the circle C could be released by a translation along the axis joining its center to the middle of a, b . However, there is another direction θ_2 such that A_{θ_2} is a triangle, yielding $w_h(B_{\theta_2}) \geq w_h(A_{\theta_2})$: indeed if the vertices a and b are at the same altitude, then choose $\theta_2 = \theta_1 + \frac{\pi}{2} \bmod \pi$. Otherwise if a is higher than b , assume that, for the value $\theta = 0$, the projection of b on V_θ is, say, on the left of the polygon of the projections of a, c, d, e . Then this projection of b is on the right for the value $\theta = \pi$. Therefore by continuity it has to cross this polygon for some $\theta_2 \in [0, \pi]$. In conclusion, the inequalities $w_h(B_{\theta_1}) \leq w_h(A_{\theta_1})$ and $w_h(B_{\theta_2}) \geq w_h(A_{\theta_2})$ yield the contradiction.

4. For general holdable convex bodies that are not icebergs, necessary conditions for equality $w = d$ can be derived from (1): the first equality $w = w_h(A_{\theta_0} \cup B_{\theta_0})$ implies that the strip measuring $w_h(A_{\theta_0} \cup B_{\theta_0})$ has to be vertical; secondly $K_0 = K \cap H$ must contain all diameters of C corresponding to the directions θ where $w_h(A_\theta) = w_h(B_\theta) = d$.

In particular, if K is a tetrahedron such that $w = d$, then by projection in H , the four edges joining each vertex of K^+ to each vertex of K^- form a rhombus: they are tangent to C and the points of tangency are on diameters, see e.g. Figure 4, right. Because the distances between these points of tangency and the vertices of the rhombus are proportional to the distances between the plane H containing C and the vertices of the tetrahedron, it follows that the two other edges, one joining the vertices of K^+ , and one joining those of K^- , are horizontal, orthogonal one to the other, and joined by their common orthogonal straight line in

their middle. To sum up, tetrahedra which satisfy $w = d$ are those with two non-intersecting orthogonal edges, joined by a common orthogonal straight line in their middle. One can see that they form a 3-dimensional submanifold of the 6-dimensional space of congruence classes of tetrahedra in \mathbb{R}^3 .

5. The case of “asymptotic equality” for Theorem 1 (a) can be described as follows. For convenience, we use the framework of Nonstandard Analysis (NSA for short) but this is not essential: instead of one nonstandard convex body K , the reader who is not acquainted with NSA may consider a whole sequence $(K_n)_{n \in \mathbb{N}}$. Then expressions such as “ $a(K)$ is i-close to b ” (notation $a(K) \simeq b$), resp. “ $a(K)$ is i-large” have to be replaced by “there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $a(K_{n_k})$ tends to b , resp. $a(K_{n_k})$ tends to $+\infty$, as $k \rightarrow +\infty$.”

If K is a convex body with $d = 2$ and w i-close to 3, then all inequalities in (2) have to be *almost equalities*, i.e. equalities up to i-small numbers. The last one $w_2(I \cap H) \simeq \frac{3d}{2}$ implies that $I \cap H$ is i-close to an equilateral triangle of height 3 (with the Hausdorff-Pompeiu distance). Here we use a well-known result due to Blaschke [5]: given a planar convex set A with planar width $w_2(A)$ and inradius $r(A)$, equality $w_2(A) = 3r(A)$ holds only if A is an equilateral triangle. We now assume that $I \cap H$ is i-close to the triangle of vertices $(-2, 0)$, $(-2j, 0)$ and $(-2j^2, 0)$ where $j = \exp(\frac{2\pi i}{3})$. We can also describe the position of points of K^+ that prevent the holding circle to escape from the body. Given any horizontal slice K_h of K^+ with circumscribing circle C_h of diameter $d_h > d$, we have that $d_h \simeq d$, that the segment joining the centers of C and C_h is almost vertical, and that all points of K_h out of the horizontal circle of diameter d and same center as C_h must project on C to points i-close to one of the three points $(1, 0)$, $(j, 0)$, $(j^2, 0)$.

6. We now discuss other ratios than $\frac{d}{w}$. Concerning the ratio $\frac{\delta}{w}$, one has the same result: this ratio, too, can be as close to $\frac{2}{3}$ as desired. To see this, we have to slightly modify the example of Figure 3, however, because this octahedron has also large holding circles. Actually, let $\alpha = \alpha(a), \beta = \beta(a) \in]0, 1[$, $A = (z_A, 0)$, $A' = (z_{A'}, 0)$ on two upper edges with $z_A = a\alpha + ja(1 - \alpha)$, $z_{A'} = a\alpha + j^2a(1 - \alpha)$ and $B = (z_B, -h)$, $B' = (z_{B'}, -h)$ on two lower edges with $z_B = -2\beta - 2j(1 - \beta)$, $z_{B'} = -2\beta - 2j^2(1 - \beta)$ be such that $z_A, z_{A'}, z_B, z_{B'}$ form a rectangle with diagonals orthogonal to the aforementioned edges (i.e. such that $\text{Im } z_A = \text{Im } z_{B'}$ and $\arg(z_A - z_B) = \frac{\pi}{3}$). Then the segments AB and $A'B'$ are diameters of a common holding circle. Nevertheless, it is possible to avoid these circles, either by moving slightly some vertices or by adding a seventh vertex, say $(0, 1)$, in such a manner that there are no other holding circles than those close to C ; in particular holding circles of this seven-vertex polyhedron have diameters less than $2a$, yielding a ratio $\frac{\delta}{w}$ close to $\frac{2}{3}$.

7. Concerning the ratio $\frac{d}{D}$, Tudor Zamfirescu presented in conferences the following “bevelled cylinder”: with $R > 0$ arbitrarily large, let $x_1 = (-R, -1, 0)$, $x_2 = (-R, 1, 0)$, $x_3 = (R, 0, -1)$ and $x_4 = (R, 0, 1)$, let C_1 and C_2 be unit circles with axis the $0x$ axis, one centered at $(1 - R, 0, 0)$ and the other at $(R - 1, 0, 0)$; then consider the convex hull of $\{x_1, x_2, x_3, x_4\} \cup C_1 \cup C_2$. One can see that any circle with diameter $[-R, R] \times \{0\} \times \{0\}$ which does not cross the interior of the body (i.e. with axis far enough from the planes $y = 0$ and $z = 0$), is a holding circle of minimal diameter, hence $d = R$ whereas $D = 1$.

8. A simple example with ratio $\frac{d}{D}$ as small as desired is the following one. Given $\varepsilon > 0$ arbitrarily small, the tetrahedron with vertices $(0, \pm\varepsilon, 0)$ and $(\pm 1, 0, 1)$ has a holding circle C with diameter $d = 2 \sin \alpha$ where α is given by $\varepsilon = \tan \alpha$. This is easily seen by orthogonal projection in a horizontal plane. This circle is horizontal, centered on the $0z$ axis at altitude $d^2/4$, see Figure 4.

One can verify that the axis of the minimal circumscribing cylinder is the straight line $\{(x, 0, \frac{1-\varepsilon^2}{2}) ; x \in \mathbb{R}\}$ and its diameter is $D = 1 + \varepsilon^2$, yielding $\frac{d}{D}$ arbitrarily small. Of course

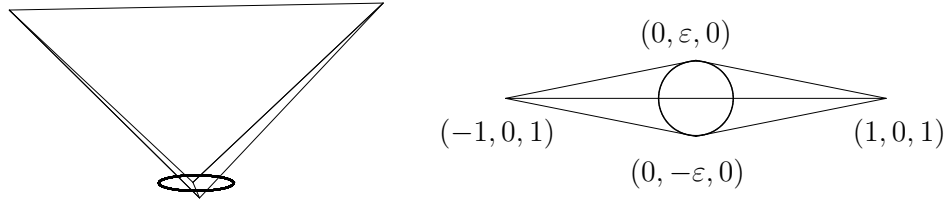


Figure 4: a *tetrahedron* with small ratio $\frac{d}{D}$; here $\varepsilon = 0.2$.

this tetrahedron has also large holding circles, e.g. any circle of diameter $[(0, 0, 0), (0, 0, 1)]$ which does not cross the interior of the tetrahedron (i.e. with axis far enough from the planes $x = 0$ and $y = 0$), but if we add the vertex $(0, 0, -\varepsilon^2)$ we obtain a five-vertex polyhedron with the same holding circle and with holding circles only close to C , thus with $\frac{\delta}{D}$ arbitrarily small.

There exist also *tetrahedra* with $\frac{\delta}{D}$ arbitrarily small, e.g. the one with vertices $A_1 = (-2, -1, \varepsilon)$, $A_2 = (-1, 0, 0)$, $A_3 = (2a, a, \varepsilon)$, $A_4 = (a, 0, 0)$ with $\varepsilon > 0$ arbitrarily small and $a = \varepsilon^2$. It is easy to verify that the circle in the plane $x = 0$ of diameter ε and center $(0, 0, \frac{\varepsilon}{2})$ is actually a holding circle. Indeed, the edges A_1A_3 and A_2A_4 have the axis $0z$ as a common orthogonal straight line and the edges A_1A_4 and A_2A_3 cross the plane $x = 0$ at $(0, -\frac{a}{a+2}, \frac{a\varepsilon}{a+2})$ and $(0, \frac{a}{2a+1}, \frac{\varepsilon}{2a+1})$, hence in the interior of the circle since $a < 2\varepsilon^2$. One can also verify that holding circles must be close to the origin and of small diameter.

9. As a conclusion, we briefly describe related topics of the literature. The first one is the problem of immobilization of convex bodies, a notion first introduced by W. Kuperberg in [15]. In [10], Czyzowicz, Stojmenovic and Urrutia prove that two-dimensional convex figures — except circular disks — can be immobilized by at most four points. In [6], Bracho, Montejano and Urrutia show that three points suffice for convex figures bounded by a curve of class \mathcal{C}^2 . Mayer gives additional results and extensions in [17]. The three-dimensional case is studied by Bracho, Mayer, Fetter and Montejano in [7]: a necessary condition for four points to immobilize a \mathcal{C}^2 convex body is that the four normal lines belong to one ruling of a quadratic surface. These questions of immobilization are motivated by grasping problems in robotics, see e.g. [18, 19].

A second related problem is to look for convex bodies passing through holes. Zindler [26] already considered an affine image of a cube passing through fairly small holes. In [12], Itoh and Zamfirescu look for the shape of a hole of minimal diameter and width through which can pass the regular unit tetrahedron T . They find a hole of diameter $\sqrt{3}/2$, the width of a face of T , and of width $\sqrt{2}/2$, the width of T . In [13], Itoh, Tanoue and Zamfirescu study the same tetrahedron passing through a circular and a square hole. Triangular holes are considered by Bárány, Maehara and Tokushige in [1] and higher dimensional holes in [16].

Another related topic is known as Prince Rupert's problem, see [9], problem B4. The original question is to cut a hole in the unit cube, large enough to let a larger cube passing through it. See also [14] for generalizations to rectangles.

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